Math 208 Midterm II

November 17, 2023

NAME:

Section: 10:30 or 11:30 (circle the one you are registered in)

UW EMAIL:

1	/20
2	/20
3	/20
4	/20
5	/20
Total	/100

Instructions. [1]

- **Please write your initials in the top right hand corner of each page.**
- For each problem below give a carefully explained solution using the vocabulary and notation from class. A correct answer with no supporting work or explanation will receive a zero.
- If a solution involves a numerical answer, collect all terms and reduce all fractions. Put a $\boxed{\text{box}}$ around your final answers.
- You are allowed a simple calculator and notesheet. Other notes, electronic devices, etc are not allowed. Take a few pencils from your pencil case out and put all other items away for the duration of the exam.
- All the questions can be solved using (at most) simple arithmetic. (If you find yourself doing complicated calculations, there might be an easier solution...)
- This exam is printed doubled sided. The last page is intentionally blank. You can use this as scratch paper or for more room for your solutions, but please label your work clearly if you intend for us to grade it.
- Raise your hand if you have any questions or spot a possible error.

Good luck!

[1] Test code: 8742

(1) Let $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^4$ be a function defined by $T(x, y, z) = (2x+z, 2y+3z, x+z, 0)$.

(a) (5 pts) What is $T(2, 2, 4)$?

$T(2,2,4) = \begin{bmatrix} 8 \\ 16 \\ 6 \\ 0 \end{bmatrix}$ or $(8,16,6,0)$. $T(3,2,4) = \begin{bmatrix} 10 \\ 16 \\ 7 \\ 0 \end{bmatrix}$ $T(4,2,4) = \begin{bmatrix} 12 \\ 16 \\ 8 \\ 0 \end{bmatrix}$ etc.

(b) (5 pts) Is T a linear transformation? Justify your answer by either finding a matrix that goes with it or by proving it does not satisfy the definition of a linear transformation.

Yes $T(x,y,z) = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 3 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$T: \mathbb{R}^3 \to \mathbb{R}^4$

(c) (10 pts) Is T one-to-one, onto, both, or neither? Justify your answer.

Why 1-1: Rank of matrix is 3

Cols are linearly independent

Since $A \sim B$ and

B has a pivot in every column

$A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 3 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix} = B$

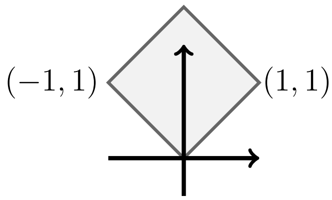
$R_3 - \frac{1}{2}R_1$

Why not onto: rank$(A) < 4$, one row of B does not have a pivot. $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \notin \text{col}(A) = \text{range}(T)$, etc.

(2) For each of the regions below, find a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ that takes the unit square

$$\mathcal{U} = \{(x, y) \in \mathbb{R}^2 \ : \ 0 \le x \le 1 \text{ and } 0 \le y \le 1\}$$

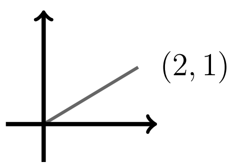
to the given region, or explain why it's impossible to do so. (Each 10 pts)

(a) 

Let $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ and $T(x,y) = A\begin{bmatrix} x \\ y \end{bmatrix}$

Yes! It's possible. Then $T(0,0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $T(1,0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$
$T(0,1) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $T(1,1) = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$.

By linearity, this suffices to prove the whole shaded square is the image of the unit square under the transformation $\begin{bmatrix} x \\ y \end{bmatrix} \to A\begin{bmatrix} x \\ y \end{bmatrix}$.

(b) 

(the line segment from $(0, 0)$ to $(2, 1)$)

Yes! It's possible. Let $B = \begin{bmatrix} 1 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$

Then $B\begin{bmatrix} x \\ y \end{bmatrix}$ is a linear transformation
such that $B\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $B\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix}$, $B\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix}$
and $B\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix} + \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

For any pt. in $\mathcal{U}$ say $\begin{bmatrix} s \\ t \end{bmatrix}$ with $0 \le s, t \le 1$, we have $\begin{bmatrix} s \\ t \end{bmatrix} \in \mathcal{U}$ and $B\begin{bmatrix} s \\ t \end{bmatrix} = \begin{bmatrix} s+t \\ \frac{1}{2}(s+t) \end{bmatrix} = \left(\frac{s+t}{2}\right)\begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $0 \le \frac{s+t}{2} \le 1$.

So the whole unit square maps to the line segment $\left\{ t\begin{bmatrix} 2 \\ 1 \end{bmatrix} : 0 \le t \le 1 \right\}$.

(3) Let $S = \{(a, b, c) \mid a, b, c \text{ are integers}\}$. Recall the set of integers includes the whole numbers like $0, 1, 2, 3, \ldots$ and their negatives, but not real numbers with decimal parts like 1.253.

(a) (7pts) Is S closed under addition? Justify your answer.

Yes, if $\begin{bmatrix} a \\ b \\ c \end{bmatrix}, \begin{bmatrix} d \\ e \\ f \end{bmatrix} \in S$ then a, b, c, d, e, f are integers.

So $\begin{bmatrix} a \\ b \\ c \end{bmatrix} + \begin{bmatrix} d \\ e \\ f \end{bmatrix} = \begin{bmatrix} a+d \\ b+e \\ c+f \end{bmatrix} \in S$ since the sum of any two integers is an integer.

(b) (7pts) Is S closed under scalar multiplication? Justify your answer.

No, consider $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \in S$ and $1.7 \in \mathbb{R}$

then $1.7 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1.7 \\ 0 \\ 0 \end{bmatrix} \notin S$ since 1.7 is not an integer.

(c) (6pts) Is S a subspace? Justify your answer.

No. A subspace must be closed under scalar multiplication by any real number.

(4) Let $M = \begin{bmatrix} 0 & 4 & 8 \\ 1 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

(a) (10pts) Compute M^{-1} using Gauss-Jordan elimination and verify your answer by matrix multiplication.

$$\left[\begin{array}{ccc|ccc} 0 & 4 & 8 & 1 & & \\ 1 & 4 & 0 & & 1 & \\ 0 & 0 & 2 & & & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 4 & 0 & 0 & 1 & 0 \\ 0 & 4 & 8 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 4 & 0 & 0 & 1 & 0 \\ 0 & 4 & 0 & 1 & 0 & -4 \\ 0 & 0 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 4 \\ 0 & 4 & 0 & 1 & 0 & -4 \\ 0 & 0 & 2 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & & & -1 & 1 & 4 \\ & 1 & & \frac{1}{4} & 0 & -1 \\ & & 1 & 0 & 0 & \frac{1}{2} \end{array}\right] \text{ so } \boxed{M^{-1} = \begin{bmatrix} -1 & 1 & 4 \\ \frac{1}{4} & 0 & -1 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}}$$

Check: $MM^{-1} = \begin{bmatrix} 0 & 4 & 8 \\ 1 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} -1 & 1 & 4 \\ \frac{1}{4} & 0 & -1 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 4/4 & 0 & -4+8/2 \\ -1+4/4 & 1 & 4-4 \\ 0 & 0 & 2/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

(b) (5pts) Use your answer in Part (a) to find all possible solutions to the matrix equation

$$M \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \\ 40 \end{bmatrix}.$$

$$M^{-1}\begin{bmatrix} 4 \\ 8 \\ 40 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 4 \\ \frac{1}{4} & 0 & -1 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 4 \\ 8 \\ 40 \end{bmatrix} = \begin{bmatrix} -4+8+160 \\ 1-40 \\ 20 \end{bmatrix} = \begin{bmatrix} 164 \\ -39 \\ 20 \end{bmatrix}$$

(c) (5pts) What linear combination(s) of the columns of M equals $\begin{bmatrix} 4 \\ 8 \\ 40 \end{bmatrix}$?

check

$$\boxed{164\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - 39\begin{bmatrix} 4 \\ 4 \\ 0 \end{bmatrix} + 20\begin{bmatrix} 8 \\ 0 \\ 2 \end{bmatrix}} = \begin{bmatrix} -39\cdot 4+160 \\ 164-39\cdot 4 \\ 40 \end{bmatrix}$$

$39\cdot 4 = 156$

$$= \begin{bmatrix} 4 \\ 8 \\ 40 \end{bmatrix} \checkmark$$

(5) Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be an invertible matrix.

(a) (5 pts) What is A^{-1} ? (no justification needed)

$A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ by formula from book + lecture

(b) (5 pts) What is A^T? (no justification needed)

$A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$ by definition

(c) (5 pts) If $A^{-1} = A^T$ and $Det(A) = 1$, what equations must a, b, c, d satisfy?

$Det(A) = ad-bc$

so $\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \Rightarrow$ $a = d$ ① $-b = c$ ②

and since $ad-bc=1 \Rightarrow a^2+b^2=1$ ③.

(d) (5pts) If $A^{-1} = A^T$, do we always have $Det(A) = 1$? Justify your answer.

No! $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = A^T = A^{-1}$

Check: $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

and $Det(A) = -1$ by usual formula for 2x2 determinants.

See steps to characterize all 2x2 matrices A s.t. $A^{-1} = A^T$ below.

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Step 1

$Det(A^T) = ad-bc = Det(A)$. by formula for Det of a 2x2 matrix.

If $A^{-1} = A^T$ then $Det(A^{-1}) = Det(A^T)$.

Compute $Det(A^{-1}) = Det \begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix} = \frac{ad}{(ad-bc)^2} - \frac{(-b)(-c)}{(ad-bc)^2}$

$$= \frac{ad-bc}{(ad-bc)^2} = \frac{1}{ad-bc} = \frac{1}{Det(A)}.$$

So $Det(A^{-1}) = Det(A^T) = Det(A) \Rightarrow$
$\frac{1}{ad-bc} = ad-bc. \Rightarrow 1 = (ad-bc)^2 = Det(A)^2$

$\Rightarrow Det(A)^2 - 1 = 0$ and we can solve this quadratic equation for $Det(A) = \pm 1$

Step 2

Consider the case $Det(A) = -1$, then

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$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} -d & b \\ c & -a \end{bmatrix}.$$

So $A^T = A^{-1} \Rightarrow \begin{bmatrix} -d & b \\ c & -a \end{bmatrix} = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$

So $\mathrm{Det}(A) = -1 \Rightarrow \boxed{a = -d,\ b = c}$

So $ad - bc = -a^2 - b^2 = -1$

or $\boxed{a^2 + b^2 = 1}$ again.

Conclusion: theorem:

$$\left\{ \begin{bmatrix} a & b \\ b & -a \end{bmatrix} : \begin{matrix} a, b \in \mathbb{R} \\ a^2+b^2=1 \end{matrix} \right\} \cup \left\{ \begin{bmatrix} a & b \\ -b & a \end{bmatrix} : \begin{matrix} a, b \in \mathbb{R} \\ a^2+b^2=1 \end{matrix} \right\}$$

$$= \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^T \right\}.$$